

# Property (T) and the Furstenberg Entropy of Nonsingular Actions

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## Abstract

We establish a new characterization of property (T) in terms of the Furstenberg entropy of nonsingular actions. Given any generating measure  $\mu$  on a countable group  $G$ , A. Nevo showed that a necessary condition for  $G$  to have property (T) is that the Furstenberg  $\mu$ -entropy values of the ergodic, properly nonsingular  $G$ -actions are bounded away from zero. We show that this is also a sufficient condition.

## 1 Introduction

A measurable action on a probability space  $G \curvearrowright (X, \eta)$  is called a *nonsingular action* if the measure  $\eta$  is quasi-invariant with respect to any  $g \in G$ ; that is, if  $\eta$  and  $g_*\eta$  are equivalent (i.e., mutually absolutely continuous) measures. We say that the action is *properly nonsingular* if  $\eta$  is not equivalent to a  $G$ -invariant probability measure.

A probability measure  $\mu$  on a countable group  $G$  is *generating* if its support generates  $G$  as a semigroup.

Let  $G \curvearrowright (X, \eta)$  be a nonsingular action and let  $\mu$  be a probability measure on  $G$ . The *Furstenberg entropy* [5] or  $\mu$ -*entropy* is defined by

$$h_\mu(X, \eta) = \sum_{g \in G} \mu(g) \int_X -\log \frac{dg_*^{-1}\eta}{d\eta}(x) d\eta(x).$$

Jensen's inequality implies that  $h_\mu(X, \eta) \geq 0$  and that, for generating measures, equality holds if and only if  $\eta$  is an invariant measure. Furstenberg entropy is an important conjugacy invariant of nonsingular actions [11], and in particular of *stationary* actions;  $G \curvearrowright (X, \eta)$  is  $\mu$ -stationary if  $\mu * \eta = \eta$ .

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The pair  $(G, \mu)$  is said to have an *entropy gap* if there exists some constant  $\epsilon = \epsilon(G, \mu) > 0$  such that the  $\mu$ -entropy of any ergodic, properly nonsingular  $G$ -action on a probability space is at least  $\epsilon$ .

A group  $G$  has *property (T)*, if any unitary representation of  $G$  which has almost invariant vectors admits a non-zero invariant vector. The purpose of this paper is to establish a similar characterization of property (T), using nonsingular actions instead of unitary representations, and thinking of those with small entropy values as being “almost invariant”.

**Theorem 1.** *Let  $G$  be a countable group. Then  $G$  has property (T) if and only if for every (equivalently, for some) generating measure  $\mu$ ,  $(G, \mu)$  has an entropy gap.*

For the case that  $G$  has property (T) this theorem follows from Nevo [10, Propositions 4.1, 4.2]. Note that the statements of these propositions omit the requirements that  $\eta$  is ergodic and properly nonsingular; the result is no longer true without either of these hypotheses. For completeness, we include in Appendix A a reproduction of Nevo’s proof of this direction.

The idea of the proof of the other direction (i.e., that groups without property (T) have no entropy gap) is the following. We first consider the group  $2_{fin}^{\mathbb{N}}$ ; the elements of this group are the finite subsets of  $\mathbb{N}$  and the group operation is symmetric difference. For this group, we construct in Section 2 very simple nonsingular actions which show that  $2_{fin}^{\mathbb{N}}$  has no entropy gap, for a family of natural measures on  $2_{fin}^{\mathbb{N}}$ .

Then, in Section 3, we observe that any countable group  $G$  without property (T) admits a non-trivial cocycle into  $2_{fin}^{\mathbb{N}}$ . Using these cocycles one can construct skew-product  $G$ -actions over  $2_{fin}^{\mathbb{N}}$  spaces; the entropy formula for skew-product actions is established in Section 4. Finally, in Section 5 we prove Theorem 1, by showing that the  $2_{fin}^{\mathbb{N}}$ -actions that we construct in Section 2, and which have arbitrarily low Furstenberg entropy, can be lifted to skew-product  $G$ -actions with arbitrarily low Furstenberg entropy.

## 1.1 Related literature

### 1.1.1 Characterizing property (T)

Recently, Ozawa [12] showed that property (T) can be characterized by the spectral gap of  $\Delta_\mu \in \mathbb{R}[G]$ , the Laplacian associated with a finitely supported, symmetric generating measure  $\mu$ . This characterization is one of many which have been established since Kazhdan’s first definition of property (T) [9], some of which have been instrumental in proving property (T) for groups that were previously not known to have it. We refer the reader to Bekka, de La Harpe and Valette [1] for a complete discussion.

### 1.1.2 Stationary actions

A particularly interesting class of nonsingular actions are the stationary ones. Furstenberg entropy has been a useful tool in the study of stationary actions (e.g., [5, 8]), and the study of the set of entropy values realizable by properly nonsingular stationary actions has attracted some interest [11, 2, 6].

Nevo's theorem (Theorem A.1) implies that for every  $(G, \mu)$ , where  $G$  has property (T), the Furstenberg entropy values of properly nonsingular, ergodic  $\mu$ -stationary actions are bounded away from zero; in this case we say that  $(G, \mu)$  has a *stationary entropy gap*. In [2] and [6] it is shown that some  $(G, \mu)$  without property (T) have no stationary entropy gap. However, this is not a characterization of property (T): in a previous paper [3, Proposition 7.5] we show that there exist  $(G, \mu)$  without property (T), but with a stationary entropy gap.

A question that therefore remains open is that of characterizing the pairs  $(G, \mu)$  that have a stationary entropy gap. More narrowly, it is not even known that all amenable groups have no stationary entropy gap.

## 2 The case of $2_{fin}^{\mathbb{N}}$

Let  $2_{fin}^{\mathbb{N}}$  denote the set of all finite subsets of the natural numbers  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Endowed with the operation of symmetric difference,  $2_{fin}^{\mathbb{N}}$  is an abelian group in which every element other than the identity is an involution. For  $T \in 2_{fin}^{\mathbb{N}}$ , let  $\max(T) = \max_{t \in T} t$ . We also let  $\max(\emptyset) = 0$ . Because  $2_{fin}^{\mathbb{N}}$  is an abelian group we will use additive notation when expressing group multiplication. Thus  $T + S := T \Delta S = (T \setminus S) \cup (S \setminus T)$  for any  $T, S \in 2_{fin}^{\mathbb{N}}$ .

Let  $2_{fin}^{\mathbb{N}}$  act on  $2^{\mathbb{N}}$  by symmetric difference, identifying elements of  $2^{\mathbb{N}}$  with subsets of  $\mathbb{N}$ . Let  $\omega_p$  be a measure on  $2^{\mathbb{N}}$  given by  $\omega_p = \prod_{\mathbb{N}} B(p)$ , where  $B(p)$  is the Bernoulli  $p$  measure on  $\{0, 1\}$ .

**Proposition 2.1.** *Consider the action  $2_{fin}^{\mathbb{N}} \curvearrowright (2^{\mathbb{N}}, \omega_p)$ . Then, for any probability measure  $\mu$  on  $2_{fin}^{\mathbb{N}}$ , the  $\mu$ -entropy of  $(2^{\mathbb{N}}, \omega_p)$  is*

$$h_{\mu}(2^{\mathbb{N}}, \omega_p) = \varphi(p) \sum_{T \in 2_{fin}^{\mathbb{N}}} \mu(T) |T| \leq \varphi(p) \sum_{T \in 2_{fin}^{\mathbb{N}}} \mu(T) \max(T),$$

where  $\varphi(p) = \log \frac{p}{1-p} (2p - 1)$ .

Note that  $\lim_{p \rightarrow 1/2} \varphi(p) = 0$  and that for  $p \neq 1/2$  it holds that  $\varphi(p) > 0$ .

*Proof.* For  $T \in 2_{fin}^{\mathbb{N}}$ ,  $T_*^{-1} \omega_p = T_* \omega_p = \prod_{n \in T} B(1-p) \prod_{n \notin T} B(p)$ . Hence

$$\int_X -\log \frac{dT_* \omega}{d\omega}(x) d\omega(x) = \varphi(p) |T|$$

where  $\varphi(p) = \log \frac{p}{1-p} (2p - 1)$  is the Kullback-Leibler divergence between  $B(p)$  and  $B(1-p)$ . Finally, it follows that for any measure  $\mu$  on  $2_{fin}^{\mathbb{N}}$ ,

$$h_{\mu}(2^{\mathbb{N}}, \omega_p) = \varphi(p) \sum_{T \in 2_{fin}^{\mathbb{N}}} \mu(T) |T| \leq \varphi(p) \sum_{T \in 2_{fin}^{\mathbb{N}}} \mu(T) \max(T).$$

□

It is easy to show that  $\omega_p$  is ergodic and is not equivalent to an invariant probability measure (see Lemma 4.2; an invariant measure  $\lambda$  would have to satisfy  $\lambda(\{x : x_n = 1\}) = 1/2$ ). Hence it follows from this proposition that  $2_{fin}^{\mathbb{N}}$  does not have an entropy gap, for any  $\mu$  with  $\sum_{T \in 2_{fin}^{\mathbb{N}}} \mu(T) \max(T) < \infty$ . In the remainder of this paper we lift this result to any group without property (T).

### 3 Non-property (T) and cocycles

Before stating the next proposition, let us recall some standard definitions. Given a nonsingular action  $G \curvearrowright (X, \eta)$  and a countable group  $\Gamma$ , a *cocycle*  $c : G \times X \rightarrow \Gamma$  is a measurable map such that

$$c(gh, x) = c(g, hx)c(h, x)$$

for a.e.  $x$  and every  $g, h \in G$ . Two such cocycles  $c_1, c_2 : G \times X \rightarrow \Gamma$  are *cohomologous* if there exists a measurable function  $\beta : X \rightarrow \Gamma$  such that

$$c_1(g, x) = \beta(gx)c_2(g, x)\beta(x)^{-1}$$

for every  $g \in G$  and a.e.  $x$ . A cocycle is a *coboundary* if it is cohomologous to the trivial cocycle (whose essential image is contained in the trivial element  $\{e\}$ ).

**Proposition 3.1.** *Assume  $G$  does not have property (T). Let  $\rho : G \rightarrow \mathbb{N}$  be a proper function (so  $\rho^{-1}(n)$  is finite for every  $n \in \mathbb{N}$ ). Then there exists an ergodic probability measure preserving action  $G \curvearrowright (X, \eta)$  and a Borel cocycle  $c : G \times X \rightarrow 2_{fin}^{\mathbb{N}}$  such that  $c$  is not cohomologous to a cocycle with a finite image. Moreover, for every element  $g \in G$ ,*

$$\int_X \max(c(g, x)) \, d\eta(x) \leq \rho(g).$$

*Proof.* This result follows from the proof of [7, Theorem 2.1] as we now explain. It is a well-known result of Connes and Weiss [4] that if  $G$  does not have property (T) then there exists an ergodic probability measure preserving action  $G \curvearrowright (X, \eta)$  which is not strongly ergodic (see also [1, Theorem 6.3.4]). By [7, Lemma 2.4] there exist Borel sets  $D_n \subset X$  such that the following hold.

1.  $D_n$  is asymptotically invariant:  $\lim_{n \rightarrow \infty} \eta(gD_n \Delta D_n) = 0$  for every  $g \in G$ ;
2.  $\lim_{n \rightarrow \infty} \eta(D_n) = 1/2$ .
3. For every  $g \in G$  with  $\rho(g) \leq n$ ,  $\eta(D_n \Delta gD_n) < 2^{-n}$ .

To be precise, the last statement follows from an easy modification of the proof of [7, Lemma 2.4]; see equation (2.10) there.

Define a map  $\phi : X \rightarrow 2^{\mathbb{N}}$  by  $\phi(x) = \{j \in \mathbb{N} : x \in D_j\}$ . We let  $2_{fin}^{\mathbb{N}}$  act on  $2^{\mathbb{N}}$  by symmetric difference.

Let  $g \in G$ . If  $m \geq \rho(g)$  then

$$\eta(\{x \in X : \phi(gx) \notin 2_{fin}^{\mathbb{N}} \triangle \phi(x)\}) \leq \sum_{k=m}^{\infty} \eta(D_k \triangle gD_k) < 2^{-m+1}.$$

Since  $m \geq \rho(g)$  is arbitrary, the left hand side must equal zero. Therefore  $\phi(gx) \in 2_{fin}^{\mathbb{N}} \triangle \phi(x)$  for  $\eta$ -a.e.  $x \in X$ .

Define the cocycle  $c : G \times X \rightarrow 2_{fin}^{\mathbb{N}}$  by  $c(g, x) = T$  if  $\phi(gx) = T \triangle \phi(x)$ . If  $\rho(g) = m$  then

$$\begin{aligned} \int_X \max(c(g, x)) \, d\eta(x) &\leq m - 1 + \int_{\{x: \max(c(g, x)) \geq m\}} \max(c(g, x)) \, d\eta(x) \\ &\leq m - 1 + \sum_{n=m}^{\infty} \eta(\{x \in X : \max(c(g, x)) \geq n\}) \\ &\leq m - 1 + \sum_{n=m}^{\infty} \eta(D_n \triangle gD_n) \leq m - 1 + 2^{-m+1} \leq m. \end{aligned}$$

The cocycle  $c$  cannot be cohomologous to a cocycle with a finite image, since  $\phi$  is non-trivial. To be precise, for  $m \in \mathbb{N}$ , we let  $2^m < 2_{fin}^{\mathbb{N}}$  denote the subgroup consisting of all elements  $T \in 2_{fin}^{\mathbb{N}}$  with support in  $[m] := \{1, \dots, m\}$ . To obtain a contradiction, suppose there is a Borel map  $f : X \rightarrow 2_{fin}^{\mathbb{N}}$  and a cocycle  $b : G \times X \rightarrow 2^m$  such that

$$c(g, x) = f(gx) + b(g, x) + f(x)$$

for a.e.  $x \in X$  and every  $g \in G$ . Note that all elements of  $2_{fin}^{\mathbb{N}}$  are involutions, and so  $-f(x) = f(x)$ .

Let  $\psi(x) = f(x) \triangle \phi(x)$ . Observe that

$$\begin{aligned} \psi(gx) &= f(gx) \triangle \phi(gx) = f(gx) \triangle c(g, x) \triangle \phi(x) = f(gx) \triangle f(gx) \triangle b(g, x) \triangle f(x) \triangle \phi(x) \\ &= b(g, x) \triangle \psi(x) \end{aligned}$$

for a.e.  $x \in X$  and every  $g \in G$ .

Let  $\tilde{\psi}(x) = \psi(x) \setminus [m]$ ,  $\tilde{f}(x) = f(x) \setminus [m]$  and  $\tilde{\phi}(x) = \phi(x) \setminus [m]$ . Then, since  $b(g, x) \subseteq [m]$ ,

$$\tilde{\psi}(gx) = (b(g, x) \triangle \psi(x)) \setminus [m] = \tilde{\psi}(x),$$

and  $\tilde{\psi}$  is  $G$ -invariant. Because  $G \curvearrowright (X, \eta)$  is ergodic, there is an element  $R \in 2^{\mathbb{N} \setminus [m]}$  such that  $\tilde{\psi}(x) = R$  for a.e.  $x$ . Thus

$$R = \tilde{\psi}(x) = \tilde{f}(x) \triangle \tilde{\phi}(x),$$

which implies  $\tilde{\phi}(x) \triangle R = \tilde{f}(x) \in 2_{fin}^{\mathbb{N}}$  for a.e.  $x \in X$ .

Let  $D'_n = D_n$  if  $n \notin R$ , and otherwise let  $D'_n = X \setminus D_n$ , the complement of  $D_n$ . Then, for a.e.  $x \in X$ , there are only a finite number of elements  $j \in \mathbb{N} \setminus [m]$  such that  $x \in D'_j$ ; these are precisely the elements in  $\tilde{f}(x)$ .

It follows that if we let  $A_n = \{x \in X : \max(\tilde{f}(x)) \geq n\}$ , then  $\lim_n \eta(A_n) = 0$ . But  $A_n$  contains  $D'_n$ , and so  $\eta(A_n) \geq \eta(D'_n)$ , in contradiction to the fact that  $\lim_n \eta(D'_n) = 1/2$ .  $\square$

## 4 Entropy for skew-products

**Lemma 4.1.** *Let  $G$  be a countable group with a probability measure  $\mu$ . Let  $G \curvearrowright (X, \eta)$  be a probability measure preserving action, and let  $c : G \times X \rightarrow \Gamma$  be a cocycle to a countable group  $\Gamma$ . Also, let  $\Gamma \curvearrowright (W, \omega)$  be a nonsingular action on the probability space  $(W, \omega)$ . Let  $G \curvearrowright X \times W$  be the skew-product action*

$$g(x, w) = (gx, c(g, x)w).$$

Then

$$h_\mu(X \times W, \eta \times \omega) = \int_X h_{\mu_x}(W, \omega) \, d\eta(x)$$

where  $\mu_x$  is the pushforward of  $\mu$  under the map  $g \mapsto c(g, x)$ .

*Proof.*

$$\begin{aligned} h_\mu(X \times W, \eta \times \omega) &= - \sum_{g \in G} \mu(g) \int_{X \times W} \log \left( \frac{dg_*^{-1}(\eta \times \omega)}{d\eta \times \omega}(x, w) \right) d\eta(x) d\omega(w) \\ &= - \sum_{g \in G} \mu(g) \int_{X \times W} \log \left( \frac{dc(g, x)_*^{-1} \omega}{d\omega}(w) \right) d\eta(x) d\omega(w) \\ &= - \int_X \sum_{\gamma \in \Gamma} \mu_x(\gamma) \int_W \log \left( \frac{d\gamma_*^{-1} \omega}{d\omega}(w) \right) d\omega(w) d\eta(x) \\ &= \int_X h_{\mu_x}(W, \omega) \, d\eta(x). \end{aligned}$$

□

**Lemma 4.2.** *Let  $E_n = \{x \in 2^{\mathbb{N}} : x_n = 1\}$ . Let  $\nu$  be a Borel probability measure on  $2^{\mathbb{N}}$  such that  $\nu(E_n) \leq 1/2$  for infinitely many  $n \in \mathbb{N}$ , and let  $\omega_p$  be the i.i.d. Bernoulli  $p$  measure on  $2^{\mathbb{N}}$ , with  $p > 1/2$ . Then  $\nu$  is not absolutely continuous with respect to  $\omega_p$ .*

*Proof.* Let  $\{i_k\}_{k=1}^{\infty}$  be a sequence of indices such that  $\nu(E_{i_k}) < 1/2$ . For  $x \in 2^{\mathbb{N}}$ , denote

$$S_n(x) = \frac{1}{n} \sum_{k=1}^n 1_{E_{i_k}}(x),$$

and let

$$S(x) = \liminf_n S_n(x),$$

so that, clearly,  $\omega_p(\{x \in 2^{\mathbb{N}} : S(x) \neq p\}) = 0$ . But

$$\int_X S_n(x) \, d\nu(x) = \frac{1}{n} \sum_{k=1}^n \nu(E_{i_k}) \leq 1/2,$$

and so by Fatou's Lemma,

$$\int_X S(x) d\nu(x) \leq \liminf_n \frac{1}{n} \sum_{k=1}^n \nu(E_{i_k}) \leq 1/2 < p.$$

Hence  $\nu(\{x \in 2^{\mathbb{N}} : S(x) \neq p\}) > 0$ , and  $\nu$  is not absolutely continuous with respect to  $\omega_p$ .  $\square$

**Lemma 4.3.** *Let  $G \curvearrowright (X, \eta)$  be an ergodic probability measure preserving action, let  $c : G \times X \rightarrow 2_{fin}^{\mathbb{N}}$  be a cocycle, and let  $2_{fin}^{\mathbb{N}}$  act on  $2^{\mathbb{N}}$  by symmetric difference. Consider the skew-product action  $G \curvearrowright X \times 2^{\mathbb{N}}$  given by  $g(x, y) = (gx, c(g, x)y)$ .*

*Denote by  $\omega_p$  the Bernoulli  $p$  i.i.d. measure on  $2^{\mathbb{N}}$ . For any  $1/2 < p < 1$ , if there exists a  $G$ -invariant probability measure that is absolutely continuous with respect to  $\eta \times \omega_p$ , then  $c$  is cohomologous to a cocycle with a finite image.*

*Proof.* Let  $\lambda \ll \eta \times \omega_p$  be a  $G$ -invariant probability measure. Note that  $\lambda$  projects to a  $G$ -invariant probability measure on  $X$  which is absolutely continuous with respect to  $\eta$ . Because  $\eta$  is ergodic, this projection must equal  $\eta$ . Hence

$$d\lambda(x, y) = d\eta(x) d\lambda_x(y)$$

where  $\lambda = \int \delta_x \times \lambda_x d\eta(x)$  is the disintegration of  $\lambda$  over  $\eta$  (here  $\delta_x$  is the Dirac measure concentrated on  $\{x\}$ ). Also

$$\frac{d\lambda}{d\eta \times \omega_p}(x, y) = \frac{d\lambda_x}{d\omega_p}(y).$$

It follows that  $\lambda_x \ll \omega_p$ , for a.e.  $x$ . By the invariance of  $\lambda$  we have that, for every  $g \in G$ ,  $g_*\lambda = \lambda$  which implies

$$\lambda_{gx} = (g_*\lambda)_{gx} = c(g, x)_*\lambda_x$$

for a.e.  $x$ .

Let  $E_n = \{x \in 2^{\mathbb{N}} : x_n = 1\}$ , and let  $f(x) = \{n \in \mathbb{N} : \lambda_x(E_n) = 1/2\}$ . By Lemma 4.2  $f(x)$  is finite for a.e.  $x \in X$ , since  $\lambda_x \ll \omega_p$  for a.e.  $x \in X$ . Then

$$\begin{aligned} f(gx) &= \{n \in \mathbb{N} : \lambda_{gx}(E_n) = 1/2\} \\ &= \{n \in \mathbb{N} : c(g, x)_*\lambda_x(E_n) = 1/2\} \\ &= f(x), \end{aligned}$$

since for any measure  $\nu$  on  $2^{\mathbb{N}}$  and  $T \in 2_{fin}^{\mathbb{N}}$ ,  $T_*\nu(E_n) = 1 - \nu(E_n)$  if  $n \in T$  and  $T_*\nu(E_n) = \nu(E_n)$  otherwise; in any case,  $T_*\nu(E_n) = 1/2$  iff  $\nu(E_n) = 1/2$ .

By the ergodicity of  $\eta$  it follows that there exists a  $T \in 2_{fin}^{\mathbb{N}}$  such that  $f(x) = T$  for a.e.  $x \in X$ . Let  $m = \max(T)$ , and denote  $\tilde{c}(g, x) = c(g, x) \setminus \{1, \dots, m\}$ .

Let  $\psi(x) = \{n > m : \lambda_x(E_n) < 1/2\}$ . Then

$$\begin{aligned}\psi(gx) &= \{n > m : \lambda_{gx}(E_n) < 1/2\} \\ &= \{n > m : c(g, x)_* \lambda_x(E_n) < 1/2\} \\ &= \tilde{c}(g, x) \triangle \{n > m : \lambda_x(E_n) < 1/2\} \\ &= \tilde{c}(g, x) \triangle \psi(x),\end{aligned}$$

where the third equality is a consequence of the fact that  $\lambda_x(E_n) \neq 1/2$  for every  $n > m$  and a.e.  $x \in X$ . Note that  $\psi(x)$  is finite, by another application of Lemma 4.2. Hence  $\tilde{c}$  is a coboundary, and so  $c$  is cohomologous to the cocycle  $c \triangle \tilde{c}$  whose image is in  $2^m$ .  $\square$

## 5 Proof of main theorem

*Proof of Theorem 1.* By Theorem A.1 we may assume  $G$  does not have property (T). Let  $\mu$  be a generating measure on  $G$  and let  $\rho : G \rightarrow \mathbb{N}$  be a proper function such that

$$\sum_{g \in G} \mu(g) \rho(g) \leq 2.$$

By Proposition 3.1 there exists an ergodic probability measure preserving action  $G \curvearrowright (X, \eta)$  and a cocycle  $c : G \times X \rightarrow 2_{fin}^{\mathbb{N}}$  that is not cohomologous to a cocycle with a finite image. Moreover,

$$\int_X \max(c(g, x)) \, d\eta(x) \leq \rho(g)$$

for any  $g \in G$ . By Proposition 2.1 for every  $\epsilon > 0$  there exists a Bernoulli probability measure  $\omega_{p(\epsilon)}$  on  $2^{\mathbb{N}}$  such that for any probability measure  $\nu$  on  $2_{fin}^{\mathbb{N}}$ ,

$$h_\nu(2^{\mathbb{N}}, \omega_{p(\epsilon)}) \leq \epsilon \sum_{T \in 2_{fin}^{\mathbb{N}}} \nu(T) \max(T).$$

Consider the skew-product action  $G \curvearrowright X \times 2^{\mathbb{N}}$  given by  $g(x, T) = (gx, c(g, x)T)$ . By Lemma 4.1,

$$\begin{aligned}h_\mu(X \times 2^{\mathbb{N}}, \eta \times \omega_{p(\epsilon)}) &= \int_X h_{\mu_x}(2^{\mathbb{N}}, \omega_{p(\epsilon)}) \, d\eta(x) \\ &\leq \epsilon \int_X \sum_{T \in 2_{fin}^{\mathbb{N}}} \mu_x(T) \max(T) \, d\eta(x).\end{aligned}$$

Note that, by Proposition 3.1, we know that

$$\int_X \sum_{T \in 2_{fin}^{\mathbb{N}}} \mu_x(T) \max(T) \, d\eta(x) = \sum_{g \in G} \mu(g) \int_X \max(c(g, x)) \, d\eta(x) \leq \sum_{g \in G} \mu(g) \rho(g) \leq 2,$$



and so

$$\lim_{\epsilon \rightarrow 0} h_\mu(X \times 2^{\mathbb{N}}, \eta \times \omega_{p(\epsilon)}) = 0.$$

Since  $c$  is not cohomologous to a cocycle with a finite image, Lemma 4.3 implies that each measure  $\eta \times \omega_{p(\epsilon)}$  is properly nonsingular, and furthermore almost every measure in its ergodic decomposition is properly nonsingular. Since the entropy of  $\eta \times \omega_{p(\epsilon)}$  is a convex combination of the entropies of its ergodic components, it follows that there exist ergodic, properly nonsingular  $G$ -actions with arbitrarily small entropy, and therefore  $(G, \mu)$  does not have an entropy gap.  $\square$

## A Groups with property (T)

**Theorem A.1** (Nevo). *Let  $G$  be a countable group with property (T), and let  $\mu$  be a generating measure. Then  $(G, \mu)$  has an entropy gap.*

The following proof is based on Nevo's [10].

*Proof.* Let  $\bar{\mu}$  be the measure on  $G$  given by

$$\bar{\mu} = \sum_{n=0}^{\infty} 2^{-n-1} \mu^n,$$

where  $\mu^n$  is the convolution of  $\mu$  with itself  $n$  times. Then  $h_{\bar{\mu}}(X, \eta) = h_\mu(X, \eta)$  (see, e.g. [8], or [6, Section 2.8]). The advantage of  $\bar{\mu}$  is that it is supported everywhere on  $G$ .

Consider, given a nonsingular action  $G \curvearrowright (X, \eta)$ , the unitary representation  $\pi$  on  $L^2(X, \eta)$  given by

$$\pi(g)f(x) = \sqrt{\frac{dg_*\eta}{d\eta}(x)} f(g^{-1}x).$$

Note that  $\pi(g)^* = \pi(g^{-1})$ .

It is easy to check that by Jensen's inequality, for any  $g \in G$ ,

$$-2 \log \langle 1, \pi(g)1 \rangle = -2 \log \langle \pi(g^{-1})1, 1 \rangle \leq \int_X -\log \frac{dg_*^{-1}\eta}{d\eta}(x) d\eta(x). \quad (1)$$

Consider the Markov operator  $\pi(\bar{\mu}): L^2(X, \eta) \rightarrow L^2(X, \eta)$  given by  $\pi(\bar{\mu}) = \sum_{g \in G} \bar{\mu}(g) \pi(g)$ . Then (1) yields the bound  $-2 \log \|\pi(\bar{\mu})\| \leq h_{\bar{\mu}}(X, \eta)$ , by another application of Jensen's inequality.

Denote

$$\|\bar{\mu}\|_T = \sup \{ \|\pi(\bar{\mu})\| : \pi \text{ is a unitary representation of } G \text{ with no invariant vectors} \}.$$

If  $G$  has property (T) then  $\|\bar{\mu}\|_T < 1$  (see, e.g., Bekka, de La Harpe and Valette [1, Corollary 6.2.3]); here we use the fact that  $\bar{\mu}$  is supported everywhere.

When  $G \curvearrowright (X, \eta)$  is properly nonsingular ergodic action,  $L^2(X, \eta)$  has no invariant vectors (see [3, Lemma 7.2]). It therefore follows that

$$0 < -2 \log \|\bar{\mu}\|_T \leq -2 \log \|\pi(\bar{\mu})\| \leq h_{\bar{\mu}}(X, \eta) = h_{\mu}(X, \eta),$$

and  $(G, \mu)$  has an entropy gap, with  $\epsilon(\mu) = -2 \log \|\bar{\mu}\|_T$ . □

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